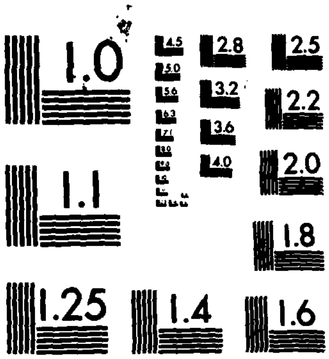

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SOME REMARKS ON THE CONTINUATION METHOD OF LERAY-SCHAUDER-RABINOWITZ AND THE METHOD OF MONOTONE ITERATIONS

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ABSTRACT

In this paper, we consider the following abstract problem. Let (E,P) be an ordered Banach space with cone P having a nonempty interior $\mathring{P}$. Let $\lambda_1,\lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, $a,b \in P$, such that $b - a \in \mathring{P}$. Let the operator $K : [\lambda_1,\lambda_2] \times [a,b] \to [a,b]$ be compact, strongly increasing with respect to the second variable for fixed $\lambda \in (\lambda_1,\lambda_2)$, strictly increasing with respect to the first variable for fixed $u \in [a,b]$. Moreover, assume that a is the only fixed point of $K(\lambda_1,\cdot)$ and that b is the only fixed point of $K(\lambda_2,\cdot)$. Consider the equation

$$(*) \qquad u = K(\lambda,u) .$$

Under the above assumptions, we prove that any closed connected subset of solutions of $(*)$ in $[\lambda_1,\lambda_2] \times [a,b]$ which meets (λ_1,a) and (λ_2,b), contains the maximal and the minimal solutions of $(*)$, which are obtained by monotone iterations. Such a subset of solutions is shown to exist. Applications to a semilinear elliptic eigenvalue problem are studied.

AMS (MOS) Subject Classifications: Primary 47H07, Secondary 47H10, 35J65

Key Words: Leray-Schauder degree, monotone iterations, nonlinear functional analysis

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SIGNIFICANCE AND EXPLANATION

Certain physical phenomena can be modelled by the nonlinear eigenvalue problem (P),

$$
\text{(P)} \qquad \begin{cases} -\Delta u = \lambda g(\cdot,u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega . \end{cases}
$$

Under minimal assumptions on the nonlinear term g, which may be of interest in some applications, the existence of solutions can be obtained by several known methods. The purpose of this paper is to compare two such methods, namely the continuation method due to Leray and Schauder as extended by Rabinowitz, and the method of monotone iterations. Our results are then applied to problem (P).

SOME REMARKS ON THE CONTINUATION METHOD OF
LERAY-SCHAUDER-RABINOWITZ AND THE METHOD OF MONOTONE ITERATIONS

Philippe Clément*

1. INTRODUCTION.

Let (E,P) be an *ordered* Banach space, see [1, p. 627]. For $a,b \in E$, $a \leq b$, $[a,b]$ denotes the order-interval $\{u \in E | a \leq u \leq b\}$.

Let $K : [a,b] \to [a,b]$ be a compact mapping, i.e. K is continuous and the range of K is relatively compact in $[a,b]$.

Since $[a,b]$ is closed and convex in E, it is a consequence of Schauder's theorem, that K possesses at least one fixed point in $[a,b]$. If K is also increasing, i.e. $u \leq v$ implies $K(u) \leq K(v)$, then the existence of a minimal (resp. maximal) fixed point of K, which we denote by $\check{u}$ (resp. $\hat{u}$), is easily established by an iteration procedure [1, p. 639].

$$\check{u} = \lim_{n\to\infty} K^{(n)}(a); \quad \hat{u} = \lim_{n\to\infty} K^{(n)}(b) .$$

If we also assume that (E,P) is normal [1, p. 627] and that P has a nonempty interior $\mathring{P}$, then $[a,b]$ is a bounded set of E, with nonempty interior $[a\mathring{,}b]$, provided that $a \ll b$, i.e. $b - a \in \mathring{P}$.

The Leray-Schauder degree of $I - K$ relative to $[a\mathring{,}b]$, $d(I - K,[a\mathring{,}b])$, see for example [6], is then well-defined, whenever K has no fixed point on the boundary of $[a,b]$ e.g. when K maps $[a,b]$ into its interior. Note that this implies

$$a \ll K(a) \quad \text{and} \quad K(b) \ll b . \tag{1.0}$$

Conversely if K satisfies (1.0), then a sufficient condition for K to map $[a,b]$ into its interior is that, K is strongly increasing, i.e. $u < v$ implies $K(u) \ll K(v)$. We shall assume K strongly increasing and satisfying (1.1). $d(I - K,[a\mathring{,}b])$ is easily computed by considering the compact homotopy:

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$$(1.1) \qquad H(t,u) := u - (1-t)c - tK(u), \quad t \in [0,1],$$
$$u \in [a,b], \text{ with } c \in [\overset{\circ}{a,b}]$$

Then, $d(I - K,[\overset{\circ}{a,b}]) = d(H(1,\cdot),[\overset{\circ}{a,b}]) = d(H(t,\cdot),[\overset{\circ}{a,b}]) = d(H(0,\cdot),[\overset{\circ}{a,b}]) = 1$, $t \in [0,1]$, by noting that the solutions (t,u) of

$$(1.2) \qquad H(t,u) = 0, \quad t \in [0,1], \quad u \in [a,b]$$

satisfy $u \in [\overset{\circ}{a,b}]$.

Since $d(H(t,\cdot),[\overset{\circ}{a,b}])$ is constant and $\neq 0$, for $t \in [0,1]$, it follows from [6, Corollaire 10, p. V - 16], that there exists a subset C of solutions of the equation (1.2) which is connected in $[0,1] \times [a,b]$ equipped with the product topology, and which meets $(0,c)$ and at least one point $(1,\bar{u})$ where $\bar{u}$ is a fixed point of K. A natural question arises, namely which fixed points $\bar{u}$ can be "reached by the homotopy" or more precisely which fixed points $\bar{u}$ of K belong to the component of $(0,c)$ in $[0,1] \times [a,b]$. In particular are $(1,\check{u})$, $(1,\hat{u}) \in C$?

In section 2, we shall prove that if $a \leq c < K(c) < \check{u}$, then the component C of $(0,c)$ in $[0,1] \times [a,b]$ meets $[1,\check{u}]$ and that $(t,u) \in C$, $0 < t < 1$ implies $u \ll \check{u}$. Similarly, one could consider the homotopy $\tilde{H}(t,u) = u - (1-t)K(u) - tc$, $t \in [0,1]$, $u \in [a,b]$. Then provided that $\hat{u} < K(c) < c \leq b$, then $\tilde{C}$, the component of $(1,c)$ meets $(0,\hat{u})$ and $(t,u) \in \tilde{C}$, $0 < t < 1$ implies $\hat{u} \ll u$.

If $\check{u} < \hat{u}$ and if there exist $u_1,u_2 \in [a,b]$ satisfying

$$(1.3) \qquad \begin{cases} \check{u} < u_1 < u_2 < \hat{u} \\ K(u_1) < u_1, \; u_2 < K(u_2) \end{cases}$$

then, Amann [2] proved that there exists a third fixed point $\bar{u}$ such that $\check{u} \ll \bar{u} \ll \hat{u}$ satisfying $\bar{u} \nleq u_1$ and $u_2 \nleq \bar{u}$. See [1, Theorem 14.2, p. 666]. Consider the homotopy:

$$(1.4) \qquad H(t,u) = \begin{cases} u - (1-2t)a - 2tK(u) & t \in [0, \tfrac{1}{2}] \\ u - 2(1-t)K(u) - (2t-1)b & t \in [\tfrac{1}{2},1] \end{cases}$$

$u \in [a,b]$, and define

$$S := \{(t,u) \in [0,1] \times [a,b] \mid H(t,u) = 0\},$$

then, if C_1 is the component of $(0,a)$ in S, we know by what preceeds that C_1 contains $(\frac{1}{2},\check{u})$; similarly, C_2 the component of $(1,b)$ in S contains $(\frac{1}{2},\hat{u})$; we shall prove in section 2, that there exists a connected set C_3 in S, which meets $(\frac{1}{2},\check{u})$, $(\frac{1}{2},\hat{u})$ and at least a third point $(\frac{1}{2},\bar{u})$ where $\bar{u}$ is a fixed point of K satisfying $\bar{u} \nleq u_1$, $u_2 \nleq \bar{u}$.

These results are special cases of Theorem 2.1 where a general homotopy

$$u = K(\lambda,u), \quad \lambda \in [\lambda_1,\lambda_2], \quad u \in [a,b] \tag{1.5}$$

is considered. There K is a compact mapping, which is strongly increasing "in u" for $\lambda \in (\lambda_1,\lambda_2)$ and strictly increasing "in λ" for $u \in [a,b]$. Then if a (resp. b) is the only fixed point of $K(\lambda_1,\cdot)$ (resp. $K(\lambda_2,\cdot)$), we prove that C the component of (λ_1,a) in $S := \{(\lambda,u) \in [\lambda_1,\lambda_2] \times [a,b] \,|\, u = K(\lambda,u)\}$ meets (λ_2,b), and contains all maximal and minimal solutions of (1.5) for $\lambda \in (\lambda_1,\lambda_2)$. Thus Theorem (2.1) relates the solutions of (1.5) obtained by applying the continuation method of Leray-Schauder-Rabinowitz [6] and the solutions of (1.5) obtained by monotone iterations [1].

In section 3, we give an application of the results of section 2 to a semilinear elliptic problem:

$$\begin{cases} -\Delta u = \lambda g(\cdot,u) & \text{in } \Omega \subset R^N \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

where we refine some results of [3], [4].

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2. THE MAIN RESULT.

Throughout this section, (E,P) denotes an ordered Banach space with cone P having a nonempty interior $\overset{\circ}{P}$, (We do not assume (E,P) to be normal), $a,b \in E$ such that $a \ll b$, and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 < \lambda_2$.

$K : [\lambda_1,\lambda_2] \times [a,b] \to [a,b]$ is continuous and has a relatively compact range in $[a,b]$ (where $[\lambda_1,\lambda_2] \times [a,b]$ is equipped with the product topology). S denotes the set of solutions of

$$u = K(\lambda,u) \qquad (\lambda,u) \in [\lambda_1,\lambda_2] \times [a,b] \ . \tag{2.1}$$

For $\lambda \in [\lambda_1,\lambda_2]$, $\hat{u}(\lambda)$ (resp. $\check{u}(\lambda)$) denotes the maximal (resp. minimal) fixed point of $K(\lambda,\cdot)$ in $[a,b]$, which are known to exist. We have

Theorem 2.1. *Let* K *defined as above satisfy the following assumptions:*

(i) *For each* $\lambda \in (\lambda_1,\lambda_2)$, $K(\lambda,\cdot)$ *is strongly increasing.*

(ii) *For each* $u \in [a,b]$, $K(\cdot,u)$ *is strictly increasing*
$(\lambda < \mu \Rightarrow K(\lambda,u) < K(\mu,u))$.

$(\mathrm{iii})_{a,b}$ a (resp. b) *is the only fixed point of* $K(\lambda_1,\cdot)$ (resp. $K(\lambda_2,\cdot)$). *Then*

(1) C *the component of* (λ_1,a) *in* S *meets* (λ_2,b).

(2) *Any closed connected set* D *in* S *which meets* (λ_1,a) *and* (λ_2,b) *contains all maximal* $\hat{u}(\lambda)$ *and minimal* $\check{u}(\lambda)$ *fixed points of* $K(\lambda,\cdot)$, $\lambda \in (\lambda_1,\lambda_2)$. *Moreover, for each* $\lambda \in (\lambda_1,\lambda_2)$,

$$\check{u}(\lambda) = \sup_{\lambda_1 < \mu < \lambda} \check{u}(\mu) = \lim_{\mu \uparrow \lambda} \check{u}(\mu) \ . \tag{2.2}$$

$$\hat{u}(\lambda) = \inf_{\lambda < \mu < \lambda_2} \hat{u}(\mu) = \lim_{\mu \downarrow \lambda} \hat{u}(\mu) \ . \tag{2.3}$$

(3) *If for some* $\bar{\lambda} \in (\lambda_1,\lambda_2)$, $\check{u}(\bar{\lambda}) < \hat{u}(\bar{\lambda})$, *and if there exist* u_1, u_2 *satisfying:*

$$\check{u}(\bar{\lambda}) < u_1 < u_2 < \hat{u}(\bar{\lambda}) \ . \tag{2.4}$$

$$u_1 > K(\bar{\lambda},u_1); \quad u_2 < K(\bar{\lambda},u_2) \ . \tag{2.5}$$

Then, any closed connected set D *in* S *which meets* $[\lambda_1,a]$ *and* $[\lambda_2,b]$ *contains a point* $[\bar{\lambda},\bar{u}]$ *where* $\bar{u} \not< u_1$, $u_2 \not< \bar{u}$.

Remark 1: If K satisfies (i), (ii) and $(iii)_a$: a is the only fixed point of $K(\lambda_1,\cdot)$, but b is not the only fixed point of $K(\lambda_2,\cdot)$, then one can apply the theorem on $[\lambda_1,\lambda_2] \times [a,\check{u}(\lambda_2)]$, provided that $a \ll \check{u}(\lambda_2)$. Indeed, $u \leq \check{u}(\lambda_2)$ implies $K(\lambda,u) \leq K(\lambda,\check{u}(\lambda_2)) \leq K(\lambda_2,\check{u}(\lambda_2)) = \check{u}(\lambda_2)$, for $\lambda \in [\lambda_1,\lambda_2]$. Then $\check{u}(\lambda_2)$ plays the role of b. Similarly, when a is not the only fixed point of $K(\lambda_1,\cdot)$ but b is for $K(\lambda_2,\cdot)$.

Remark 2: When K satisfies (i), (ii) and $(iii)_a$ and $a \leq \check{u}(\lambda_2) < b$, then from what preceeds we know that there is a connected set C_1 in $S \cap [\lambda_1,\lambda_2] \times [a,\check{u}(\lambda_2)]$ which meets (λ_1,a) and $(\lambda_2,\check{u}(\lambda_2))$. A priori the component C of $[\lambda_1,a]$ in S may not be contained in $[\lambda_1,\lambda_2] \times [a,\check{u}(\lambda_2)]$. The following lemma, which will be useful in the proof of Theorem 2.1, shows that $(\lambda,u) \in C$ implies $u \in [a,\check{u}(\lambda_2)]$.

Lemma 2.2. Let K be as in Theorem 2.1 satisfying (i), (ii), and (iv): c is a fixed point of $K(\lambda_1,\cdot)$ and d is a fixed point of $K(\lambda_2,\cdot)$ such that $c < d$.

Let C_c denote the component of (λ_1,c) in $S \cap [\lambda_1,\lambda_2] \times [a,b]$. Then for each $(\lambda,u) \in C$,

$$u \ll d \text{ holds.} \tag{2.4}$$

Similarly, let C_d denote the component of (λ_2,d) in $S \cap (\lambda_1,\lambda_2] \times [a,b]$. Then for each $(\lambda,u) \in C$,

$$c \ll u \text{ holds.} \tag{2.5}$$

Proof of Lemma 2.2.

Let $A := \{\lambda,u) \in C_c | u \leq d\}$. Then $A \neq \phi$, since $(\lambda_1,c) \in A$, moreover A is closed in C_c. For $(\lambda,u) \in A$, we have $u < d$. Otherwise $u = K(\lambda,u) < K(\lambda_2,u) \leq K(\lambda_2,d) = d$, a contradiction. Thus $u = K(\lambda,u) \ll K(\lambda,d) < K(\lambda_2,d) = d$ and $u \ll d$. This and the fact that $\lambda < \lambda_2$ for $(\lambda,u) \in A$ imply that A is open in C_c. Thus $A = C_c$ and (2.4) holds. The second part of the lemma is proved by exchanging the role of (λ_1,c) and (λ_2,d) and reversing the inequalities.

Remark. The proof of Lemma 2.2 is similar to the proof of part 2 of Theorem 1 of [4].

Proof of Theorem 2.1.

We first prove assertion (2). Let $\bar\lambda \in (\lambda_1,\lambda_2)$. We denote by $D_{\bar\lambda}$ the component of (λ_1,a) in $D \cap ([\lambda_1,\bar\lambda) \times [a,b])$. We claim that

a) $\sup_{(\lambda,u)\in D_{\bar\lambda}} \lambda = \bar\lambda$

b) $(\lambda,u) \in D_{\bar\lambda}$ implies $u < \check u(\bar\lambda)$.

First we prove a). Assume that $\sup_{(\lambda,u)\in D_{\bar\lambda}} \lambda < \bar\lambda$. Then, there exists $\bar\mu \in (\lambda_1,\bar\lambda)$ such that $D_{\bar\lambda} \cap (\{\bar\mu\} \times [a,b]) = \phi$. Set $A := D \cap (\{\bar\mu\} \times [a,b])$. $A \neq \phi$ since $\mathrm{Proj}_{[\lambda_1,\lambda_2]} D = [\lambda_1,\lambda_2]$.

If we define $C := D \cap ([\lambda_1,\bar\mu] \times [a,b])$ then C is a compact metric space, and A, $D_{\bar\lambda}$ are closed disjoint subsets of C. There is not connected set $\tilde D$ meeting both A and $D_{\bar\lambda}$, otherwise $D_{\bar\lambda} \supseteq \tilde C$ by using the maximality of $D_{\bar\lambda}$ and $D_{\bar\lambda} \cap A \neq \phi$, a contradiction. Thus by a lemma of point set topology, see for instance [6, Lemma 1.9], there are closed disjoint subsets of C, C_1 and C_2 such that $D_{\bar\lambda} \subset C_1$, $A \subset C_2$ and $C = C_1 \cup C_2$. Then define $C_3 := \{(\mu,u) \in D \mid \mu > \bar\lambda\}$. $C_2 \cup C_3$ is closed $C_1 \cap (C_2 \cup C_3) = \phi$ and $D = C_1 \cup (C_2 \cup C_3)$, contradicting the connectedness of D. Thus, $\sup_{(\lambda,u)\in D_{\bar\lambda}} \lambda = \bar\lambda$. Next we prove b). Observe that b) is a consequence of Lemma 2.2, with λ_2 replaced by $\bar\lambda$, c replaced by a and d replaced by $\check u(\bar\lambda)$. Note that $a \ll \check u(\bar\lambda)$, since $a = K(\lambda_1,a) < K(\bar\lambda,a) \le K(\bar\lambda,\check u(\lambda)) = \check u(\bar\lambda)$. Then $K(\bar\lambda,a) \ll K(\bar\lambda,\check u(\lambda))$ and $a < K(\bar\lambda,a) \ll K(\bar\lambda,\check u(\bar\lambda)) = \check u(\bar\lambda)$. This proves b). By using a), b) and the fact that the range of K is relatively compact in $[a,b]$, there exist $\bar u \in [a,b]$ and a sequence $(\lambda_n,u_n) \in D_{\bar\lambda}$ such that $\lim_{n\to\infty} \lambda_n = \bar\lambda$ and $\lim_{n\to\infty} u_n = \bar u$. By using the continuity of K, $\bar u = K(\bar\lambda,\bar u)$ and thus $\check u(\bar\lambda) \le \bar u$. But from b) it follows that $\bar u \le \check u(\bar\lambda)$, thus $\bar u = \check u(\bar\lambda)$. Since $(\lambda_n,u_n) \in D$, $(\bar\lambda,\check u(\bar\lambda)) \in D$. For the u_n we could have chosen $\check u(\lambda_n)$. Since for each sequence $\check u(\mu_n)$ such that $\mu_n \uparrow \bar\lambda$ as $n \to \infty$, there exists a subsequence which converges to $(\bar\lambda,\check u(\bar\lambda))$ we have also proven that $\lim_{\lambda\uparrow\bar\lambda} \check u(\lambda) = \sup_{\lambda<\bar\lambda} \check u(\lambda) = \check u(\bar\lambda)$. The second part of the assertion of (2) is proven in a "dual" fashion.

Next we prove assertion (1). From $(iii)_{a,b}$ and the compactness of K it follows that $\lim_{\lambda\downarrow\lambda_1} \check u(\lambda) = a$ and $\lim_{\lambda\uparrow\lambda_2} \check u(\lambda) = b$. Thus, since $b - a \in \mathring{P}$, there are $\alpha \in (0, 1/2)$ and

$\varepsilon \in (0, \lambda_2 - \lambda_1)$ such that $\hat{u}(\lambda) \le a + \alpha(b - a)$ for $\lambda \in (\lambda_1, \lambda_1 + \varepsilon)$ and $\check{u}(\lambda) \ge b - \alpha(b - a)$ for $\lambda \in (\lambda_2 - \varepsilon, \lambda_2)$. Hence $\hat{u}(\lambda) \ll \check{u}(\mu)$ for $\lambda \in (\lambda_1, \lambda_1 + \varepsilon)$ and $\mu \in (\lambda_2 - \varepsilon, \lambda_2)$. We claim that for each $\lambda \in (\lambda_1, \lambda_1 + \varepsilon)$ and $\mu \in (\lambda_2 - \varepsilon, \lambda_2)$, there is a maximal connected set $C_{\lambda,\mu}$ in $S \cap ([\lambda,\mu] \times [a,b])$ which meets $(\lambda, \hat{u}(\lambda))$ and $(\mu, \check{u}(\mu))$. For $t \in [\lambda,\mu]$, define $O_t := [K(t,a)^{\circ}, K(t,b)]$. Note that $K(t,a) \ll \check{u}(t) \le \hat{u}(t) \ll K(t,b)$, for $t \in [\lambda,\mu]$. Then $O := \bigcup_{t \in [\lambda,\mu]} \{t\} \times O_t$ is an open subset of $[\lambda,\mu] \times [a,b]$ containing no solution of (2.1) on its boundary (as a subset of $[\lambda,\mu] \times [a,b]$). We know that $d(I - K(t,\cdot), O_t) = 1$, $t \in [\lambda,\mu]$. By [7, Corollaire 10, V - 6], there exists a component $C_{\lambda,\mu}$ of $S \cap ([\lambda,\mu] \times [a,b])$ which meets $\{\lambda\} \times S_\lambda$ and $\{\mu\} \times S_\mu$ where $S_t := \{u \in [a,b] \mid (t,u) \in S\}$, $t \in [\lambda_1, \lambda_2]$. Next we prove that $C_{\lambda,\mu}$ contains $(\lambda, \hat{u}(\lambda))$ and $(\mu, \check{u}(\mu))$. We denote by c any element of $\{\lambda\} \times S_\lambda \cap C_{\lambda,\mu}$. Note that C_c, the component of (λ, c) in $S \cap ([\lambda,\mu] \times [a,b])$ is $C_{\lambda,\mu}$. Next we define $\tilde{C}_c$, the component of (λ,c) in $S \cap ([\lambda,\mu) \times [a,b])$ and as in the proof of part (2) one proves that $\sup_{(t,u) \in \tilde{C}_c} t = \mu$, and by applying the lemma 2.2, with λ_1, λ_2, d and D replaced by $\lambda, \mu, \check{u}(\mu)$ and $\tilde{C}_c$, noting that $c \le \hat{u}(\lambda) \ll \check{u}(\mu)$, we obtain $u \ll \check{u}(\mu)$ for $(t,u) \in \tilde{C}_c$. Then one chooses a sequence $(t_n, u_n) \to (\mu, \bar{u})$ such that $t_n \to \mu$, $u_n \to \bar{u}$ and one proves as in part (2) that $\bar{u} = \check{u}(\mu)$. Thus $(\mu, \check{u}(\mu))$ belongs to the closure of $\tilde{C}_c$ and hence to $C_c = C_{\lambda,\mu}$. Similarly one proves that $(\lambda, \hat{u}(\lambda)) \in C_{\lambda,\mu}$. We conclude the proof of assertion (1), by noting that $\bigcup_{\substack{\lambda \in (\lambda_1, \lambda_1 + \varepsilon) \\ \mu \in (\lambda_2 - \varepsilon, \lambda_2)}} C_{\lambda,\mu}$ is also connected and that its closure in $[\lambda_1, \lambda_2] \times [a,b]$ satisfies the requirements of assertion (1).

Finally we prove assertion (3). We define $O_1 := [\lambda_1, \bar{\lambda}) \times [a, u_1]$ and $O_2 := (\bar{\lambda}, \lambda_2] \times [u_2, b]$. O_1 and O_2 are open in $[\lambda_1, \lambda_2] \times [a,b]$. Let $\lambda \in (\lambda_1, \lambda)$, then $(\lambda, \check{u}(\lambda)) \in O_1$. Indeed,

$$a \ll \check{u}(\lambda) = K(\lambda, \check{u}(\lambda)) \le K(\bar{\lambda}, \check{u}(\lambda)) \ll K(\bar{\lambda}, u_1) \le u_1 . \tag{2.7}$$

Similarly, let $\mu \in (\bar{\lambda}, \lambda_2)$, then $(\mu, \hat{u}(\mu)) \in O_2$. We know that $(\lambda, \check{u}(\lambda)) \in D$ and $(\mu, \hat{u}(\mu)) \in D$. Since O_1 and O_2 are disjoint and D is closed and connected in

$[\lambda_1,\lambda_2] \times [a,b]$, there exists $\tilde{D}$ closed and connected in $[\lambda_1,\lambda_2] \times [a,b]$ such that

a) $\tilde{D} \subset O_1^c \cap O_2^c$

b) $\tilde{D}$ meets ∂O_1 and ∂O_2, where $O_i^c = ([\lambda_1,\lambda_2] \times [a,b])\backslash O_i$, $i = 1,2$ and ∂O_i is the boundary of O_i as subset of $[\lambda_1,\lambda_2] \times [a,b]$, $i = 1,2$.

$$\partial O_1 = [\lambda_1,\bar{\lambda}) \times \partial[a,u_1] \cup \{\bar{\lambda}\} \times [a,u_1]$$

$$\partial O_2 = (\bar{\lambda},\lambda_2] \times \partial[u_2,b] \cup \{\bar{\lambda}\} \times [u_2,b]$$

$$D \cap \partial O_1 = \{(\lambda_1,a)\} \cup B_1$$

$$D \cap \partial O_2 = \{(\lambda_2,b)\} \cup B_2$$

where

$$B_1 := \{(\lambda,u) \in D \mid \lambda = \bar{\lambda};\ u \in [a,u_1]\}$$

$$B_2 := \{\lambda,u) \in D \mid \lambda = \bar{\lambda};\ u \in [u_2,b]\}\ .$$

Note that the component of $O_1^c \cap O_2^c$ which contains (λ_1,a) is $\{(\lambda_1,a)\}$. Similarly for (λ_2,b). Thus $\tilde{D} \cap B_1 \neq \phi$, and $\tilde{D} \cap B_2 \neq \phi$. We want to prove that $\tilde{D} \cap (\{\bar{\lambda}\} \times ([a,u_1]^c \cup [u_2,b]^c)) \neq \phi$, where $[a,u_i]^c = [a,b]\backslash[a,u_i]$, $i = 1,2$. Define

$$A_1 := [\lambda_1,\bar{\lambda}] \times [a,u_1] \cup [\bar{\lambda},\lambda_2] \times [u_2\mathring{,}b]^c$$

$$A_2 := [\lambda_1,\bar{\lambda}] \times [a\mathring{,}u_1]^c \cup [\bar{\lambda},\lambda_2] \times [u_2,b]$$

Then A_1, A_2 are closed subsets of $[\lambda_1,\lambda_2] \times [a,b]$, and $(\tilde{D} \cap A_1) \cup (\tilde{D} \cap A_2) = \tilde{D}$. $\tilde{D} \cap A_1 \neq \phi$, since $\phi \neq \tilde{D} \cap B_1 \subset \tilde{D} \cap A_1$. Similarly $\tilde{D} \cap A_2 \neq \phi$. Thus

$$\tilde{D} \cap A_1 \cap A_2 \neq \phi$$

by using the connectedness of $\tilde{D}$. From the observation that $(t,u) \in \tilde{D}$ and $t < \bar{\lambda}$, $u \in [a,u_1]$ (resp. $t > \bar{\lambda}$, $[u_2,b]$) implies $u \in [a\mathring{,}u_1]$ (resp. $[u_2\mathring{,}b]$), it follows that $\tilde{D} \cap A_1 \cap A_2 = \tilde{D} \cap (\{\bar{\lambda}\}\ \chi\ ([u_2\mathring{,}b]^c \cap [a\mathring{,}u_1]^c))$. Thus there is $(\bar{\lambda},\bar{u}) \in \tilde{D} \subset D$ such that $\bar{u} \not< u_1$ and $u_2 \not< \bar{u}$. If $\bar{u} < u_1$, then $\bar{u} < \bar{u}_1$. Indeed $\bar{u} < u_1$ implies $\bar{u} = K(\bar{\lambda},\bar{u}) < K(\bar{\lambda},u_1) < u_1$. Thus $\bar{u}$ satisfies $\bar{u} \not< u_1$ and $u_2 \not< \bar{u}$. This completes the proof of the assertion (3) and of the Theorem 2.1.

3. AN EXAMPLE.

We consider the nonlinear eigenvalue problem:

$$(P) \qquad \begin{cases} -\Delta u = \lambda g(\cdot,u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary Γ.

$g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and g_u exists and is continuous. A solution of (P) is a part $(\lambda,u) \in \mathbb{R} \times W^{2,p}(\Omega)$ with $p > N$ satisfying (P). Let $\bar{u}$ be a positive, superharmonic, bounded, lower-semicontinuous function on Ω such that $g(x,\bar{u}(x)) = 0$ a.e. in Ω and such that $g(x,u) > 0$ for $0 \leq u \leq \bar{u}(x)$, $x \in \Omega$. It was shown in [4], that if S denotes the set of solutions of (P) in $\mathbb{R}^+ \times W^{2,p}$ equipped with the $\mathbb{R} \times C^1$ topology, and if C is the component of S containing $(0,0)$, then C satisfies:

1) $(\lambda,u) \in C\backslash(0,0)$ implies u is positive, superharmonic and $u(x) < \bar{u}(x)$, $x \in \Omega$.

2) for every $\lambda > 0$, C has a minimal solution $\bar{u}(\lambda)$.

3) $\lim\limits_{\lambda\to\infty} \|\bar{u}(\lambda) - \bar{u}\|_{L^p} = 0 \quad p < \infty$.

For $\lambda > 0$, we shall say that $\check{u}(\lambda)$ is the minimal (resp. maximal) solution of (P) in $[0,\bar{u}]$ if $(\lambda,\check{u}(\lambda))$ (resp. $(\lambda,\hat{u}(\lambda))$) is a solution of (P) and for any solution (λ,u) satisfying $0 \leq u(x) \leq \bar{u}(x)$, $x \in \Omega$, $u(x) \geq \check{u}(\lambda)(x)$, $x \in \Omega$ (resp. $u(x) \leq \hat{u}(\lambda)(x)$, $x \in \Omega$).

The aim of this section is to prove the following:

<u>Theorem 3.3.</u> C <u>above defined contains for each</u> $\lambda > 0$ <u>the minimal and the maximal solutions in</u> $[0,\bar{u}]$.

<u>Proof.</u> a) C contains the minimal solution $\check{u}(\lambda)$ for each $\lambda > 0$. Let $\bar{\lambda} > 0$ and $w(\bar{\lambda}) \geq 0$ be such that (3.1) $w(\bar{\lambda}) + \lambda g_u(x,u) > 0$ for $\lambda \in [0,\bar{\lambda}]$, and for $0 \leq u \leq \bar{u}(x)$, $x \in \Omega$. Then we rewrite (P) as

$$(P') \qquad \begin{cases} -\Delta u + w(\bar{\lambda})u = w(\bar{\lambda})u + \lambda g(\cdot,u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

(P') is then equivalent with

$$(P'') \qquad u = K(\lambda,u)$$

$$u \in E := \{v \in C^1(\bar\Omega) | v = 0 \text{ on } \Gamma\}$$

equipped with the C^1 norm, and

$$(3.2) \qquad K(\lambda,u)(x) := \int_\Omega G_w(x,y)[w(\bar\lambda)u(y) + \lambda g(y,u(y))]dy$$

where $G_w(\cdot,\cdot)$ denotes the Green function relative to $-\Delta u + w(\bar\lambda)u$ on Ω with Dirichlet boundary conditions.

Note that by (3.2) K is defined on $\mathbb{R} \times L^\infty(\Omega)$ and takes its values in E. In E we introduce the cone

$$P := \{u \in E | u(x) \geq 0, x \in \Omega\}$$

of positive solutions; it is standard that P has a nonempty interior $\mathring{P}$. Next we define $\tilde u(\lambda) := K(\bar\lambda,\bar u)$.

$$\tilde u(\lambda) \text{ satisfies } \begin{cases} -\Delta\tilde u(\bar\lambda) + w(\bar\lambda)\tilde u(\bar\lambda) = w(\bar\lambda)\bar u \\ \tilde u(\bar\lambda) = 0 \end{cases}$$

Then $\tilde u(\bar\lambda) \in \mathring{P}$ and $\tilde u(\lambda) < \bar u$ in Ω. By our choice of $w(\bar\lambda)$, this implies that $K(\bar\lambda,\tilde u(\bar\lambda)) < \tilde u(\bar\lambda)$. Since $K(\bar\lambda,\cdot)$ is increasing in u, $K(\bar\lambda,\cdot) : [0,\tilde u(\bar\lambda)] \to [0,\tilde u(\lambda)]$ and thus (P) has a minimal solution $\check u(\bar\lambda)$. Note that $\check u(\bar\lambda) \in \mathring{P}$. By our choice of $w(\bar\lambda)$, $u \to K(\lambda,u)$ is strongly increasing for $\lambda \in (0,\bar\lambda]$, increasing for $\lambda = 0$, and $K(\lambda,0) > 0$ for $\lambda \in (0,\bar\lambda)$ and $K(\lambda,\check u(\bar\lambda)) \ll \check u(\bar\lambda)$ for $\lambda \in (0,\bar\lambda)$. Moreover $K(\cdot,u)$ is strictly increasing in λ for each $u \in [0,\check u(\bar\lambda)]$. Thus, $K : [0,\bar\lambda] \times [0,\check u(\bar\lambda)] \to [0,\check u(\bar\lambda)]$ satisfies the assumptions of Theorem 2.1. There exists a connected set D of solutions in $\mathbb{R} \times C^1$ which meets $(0,0)$ and $(\bar\lambda,\check u(\bar\lambda))$. Since $D \subset C$, C contains $(\bar\lambda,\check u(\bar\lambda))$ and obviously, $\check u(\bar\lambda) = \bar u(\bar\lambda)$.

b) C contains the maximal solution $\hat u(\lambda)$ in $[0,\bar u]$ for each $\lambda > 0$. Let $\bar\lambda > 0$ and $w(\bar\lambda) > 0$ be chosen as in a). We denote by $S_{(\bar\lambda)} := \{u \in [0,\bar u] | (\bar\lambda,u)$ is a solution of (P). K being defined as in a), we know that $u \in S_{\bar\lambda}$ implies $u \ll K(\bar\lambda,\bar u) = \tilde u(\bar\lambda)$, and $u > 0$. Thus $S_\lambda \in [0,\tilde u(\bar\lambda)]^\circ = [0,K(\bar\lambda,\bar u)]^\circ$. Define $\lambda_n := \bar\lambda + n$, $n \in \mathbb{N}$. Then, since $\check u(\lambda_n) < \check u(\lambda_{n+1})$, $n \in \mathbb{N}$ (easily verified), we have $K(\lambda,\check u(\bar\lambda_n)) \ll K(\bar\lambda,\check u(\lambda_{n+1}))$. Moreover $K(\bar\lambda,\bar u) = \lim_{n\to\infty} K(\bar\lambda,\check u(\lambda_n))$ follows from statement 3) before Theorem 3.1. We claim that

$[0,\mathring{K}(\bar\lambda,\bar u)] = \bigcup_{n=1}^{\infty} [0,K(\mathring{\bar\lambda},\check u(\lambda_n))]$. Indeed let $v \in [0,\mathring{K}(\bar\lambda,\bar u)]$. By definition, there is $\alpha > 0$ such that $K(\bar\lambda,\bar u) - v \geq \alpha e$ where e is an element of $\mathring{P}$. Moreover $K(\bar\lambda,\bar u) = \lim_{n\to\infty} K(\bar\lambda,\check u(\lambda_n))$ in C^1 implies the existence of a sequence $\{\beta_n\}$ with $\lim_{n\to\infty} \beta_n = 0$ such that

$$K(\bar\lambda,\bar u) - K(\bar\lambda,\check u(\lambda_n)) \leq \beta_n e, \quad n \in \mathbb{N} .$$

Thus $K(\bar\lambda,\check u(\lambda_n)) - v \geq (\alpha - \beta_n)e$, $n \in \mathbb{N}$ and there are $N \in \mathbb{N}$ and $c > 0$, such that

$$K(\bar\lambda,\check u(\lambda_N)) - v \geq ce .$$

Thus $v \in [0,K(\mathring{\bar\lambda},\check u(\lambda_N))] \subset \bigcup_{n=1}^{\infty} [0,K(\mathring{\bar\lambda},\check u(\lambda_n))]$. Hence $S_{\bar\lambda} \subset \bigcup_{n=1}^{\infty} [0,K(\mathring{\bar\lambda},\check u(\lambda_n))]$ Next we observe that S_λ is compact in C^1. Hence there is $m \in \mathbb{N}$, such that

$$S_{\bar\lambda} \subset \bigcup_{n=1}^{m} [0,K(\mathring{\bar\lambda},\check u(\lambda_n))] \subset [0,K(\bar\lambda,\check u(\lambda_m))] .$$

Note that $\bar v := K(\bar\lambda,\check u(\lambda_m))$ satisfies

$$\begin{cases} -\Delta\bar v + w(\bar\lambda)\bar v = w(\bar\lambda)\check u(\lambda_m) + \bar\lambda g(\cdot,\check u(\lambda_m)) & \text{in } \Omega \\ \bar v = 0 & \text{on } \Gamma . \end{cases}$$

and $\check u(\lambda_m)$ satisfies

$$\begin{cases} -\Delta\check u(\lambda_m) + w(\bar\lambda)\check u(\lambda_m) = w(\bar\lambda)\check u(\lambda_m) + \lambda_m g(\cdot,\check u(\lambda_m)) & \text{in } \Omega \\ \check u(\lambda_m) = 0 & \text{on } \Gamma . \end{cases}$$

Since $\lambda_m > \bar\lambda$ and $g(\cdot,\check u(\lambda_m)) > 0$, we have

$$\begin{cases} -\Delta(\bar v - \check u(\lambda_m)) + w(\bar\lambda)(\bar v - \check u(\lambda_m)) \leq 0 & \text{in } \Omega \\ \bar v - \check u(\lambda_m) = 0 & \text{on } \Gamma . \end{cases}$$

Thus $\bar v < \check u(\lambda_m)$ and

$$S_{\bar\lambda} \subset [0,\check u(\lambda_m)] .$$

Next, choosing $w(\lambda_m) \geq 0$ such that

$$w(\lambda_m) + \lambda g_u(x,u) > 0 \quad \text{for} \quad \lambda \in [0,\lambda_m] ,$$

$0 \leq u \leq \bar u(x)$, $x \in \Omega$, one defines K as in a) and verifies that with this choice of

w, K satisfies the assumptions of Theorem 2.1 on $[0,\lambda_m] \times [0,\check{u}(\lambda_m)]$. Then, there is a connected set D of solutions of (P) in $\mathbb{R} \times C^1$ which contains the maximal solution $\hat{u}(\bar{\lambda})$ in $[0,\check{u}(\lambda_m)]$. But since $S_{\bar{\lambda}} \subset [0,\check{u}(\lambda_m)]$, $\hat{u}(\bar{\lambda})$ is the maximal solution in $[0,\bar{u}]$. Since $D \subset C$, C contains the maximal solution $\hat{u}(\lambda)$ of (P) in $[0,\bar{u}]$. This completes the proof of Theorem 3.1.

Remark 1. It is also a consequence of the proof that if we denote by C_λ the component of solutions of (P) in $[0,\lambda) \times C^1$ which contains $(0,0)$, then $C = \bigcup_{\lambda>0} C_\lambda$.

Remark 2. In the "bifurcation case", i.e., when g satisfies $g(x,0) = 0$, $x \in \Omega$ but $g_u(x,0) > 0$, $x \in \Omega$, then a similar analysis shows that $C = \bigcup_{\lambda>\lambda_1} C_\lambda$ when C is the component of positive solutions "emanating" from (λ_1,o), bifurcation point. Then for $\lambda > \lambda_1$, C contains all maximal solutions in $[0,\bar{u}]$. Note that in this case the minimal solution in $[0,\bar{u}]$ is 0, but it is shown in [3], [4], that C possesses a minimal solution for each $\lambda > \lambda_1$.

REFERENCES

1. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976), 620-709.
2. H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, J. Functional Analysis, 11 (1972), 348-384.
3. Ph. Clément and L. A. Peletier, Sur les solutions superharmoniques de problèmes aux valeurs propres elliptiques, C. R. Acad. Sc. Paris, T. 293 (2 Nov. 1981).
4. Ph. Clément and L. A. Peletier, On positive superharmonic solutions to semi-linear elliptic eigenvalue problems (to appear in J. Math. Anal. Appl.).
5. P. H. Rabinowitz, Some apsects of nonlinear eigenvalue problems, Rocky Mt. J. Math., 3 (1973), 161-202.
6. P. H. Rabinowitz, "Théorie du degré topologigue et applications à des problèmes aux limites non linéaires", Université Paris VI et CNRS, 1975, (notes rédigées par H. Berestycki).
7. P. H. Rabinowitz, Pairs of positive solutions of nonlinear elliptic partial differential equations, Indiana Univ. Math. J., 23 (1973), 173-186.

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In this paper, we consider the following abstract problem. Let (E,P) be an ordered Banach space with cone P having a nonempty interior $\mathring{P}$. Let $\lambda_1,\lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, $a,b \in P$, such that $b - a \in \mathring{P}$. Let the operator $K : [\lambda_1,\lambda_2] \times [a,b] \to [a,b]$ be compact, strongly increasing with respect to the second variable for fixed $\lambda \in (\lambda_1,\lambda_2)$, strictly increasing with respect

(continued)

ABSTRACT (continued)

to the first variable for fixed $u \in [a,b]$. Moreover, assume that a is the only fixed point of $K(\lambda_1,\cdot)$ and that b is the only fixed point of $K(\lambda_2,\cdot)$. Consider the equation

$$(*) \qquad u = K(\lambda,u) .$$

Under the above assumptions, we prove that any closed connected subset of solutions of $(*)$ in $[\lambda_1,\lambda_2] \times [a,b]$ which meets (λ_1,a) and (λ_2,b), contains the maximal and the minimal solutions of $(*)$, which are obtained by monotone iterations. Such a subset of solutions is shown to exist. Applications to a semilinear elliptic eigenvalue problem are studied.